FMDB *Transactions on Sustainable Social Sciences Letters*

Multivariate Modelling of Returns on Crude Oil Price Benchmarks: A Diagonal BEKK-GARCH Approach Emphasizing Positive Definiteness

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Abstract: The study examines volatility and return dynamics of four global crude oil benchmarks – Crude Oil Average (COA), Brent (COB), Dubai (COD), and West Texas Intermediate (COWTI) – denominated in Nigerian naira per barrel from January 1982 to April 2023. Monthly price data comprising 1,984 observations were obtained from the Energy Information Administration website. Using EViews 10, the study employed primary diagnostics such as descriptive statistics, unit root tests, correlation analysis, and the Jarque-Bera test to assess normality. Log transformation and performance were applied to ensure stability and detect volatility. The results showed that all price series were right-skewed and leptokurtic, while the return series were negatively skewed and significantly leptokurtic, indicating fat tails and non-normality. The presence of ARCH effects required the application of a multivariate GARCH framework. A diagonal BEKK-GARCH model was applied to capture time-varying volatility and correlations between crude oil benchmark indices. The model confirmed volatility clustering, using specific positive variance-covariance matrices. This suggests that, using specific positive matrices, the risk implications between different crude oil benchmarks (Brent, WTI, Dubai, and Mediterranean) are quantified and predictable. This provides a solid basis for the design of sovereign wealth funds and Hedging strategies to protect the Nigerian economy from global fluctuations in crude oil prices.

Keywords: Crude Oil Average; Oil-Dependent Economies; Benchmark Indicators; Co-Movement and Volatility; Oil Market Stability; Macroeconomic Planning; Globally Traded Commodity.

Received on: 02/07/2024, **Revised on:** 21/09/2024, **Accepted on:** 19/11/2024, **Published on:** 09/03/2025

Journal Homepage: https://www.fmdbpub.com/user/journals/details/FTSSSL

DOI: https://doi.org/10.69888/FTSSSL.2025.000409

Cite as: D. Z. Dum, "Multivariate Modelling of Returns on Crude Oil Price Benchmarks: A Diagonal BEKK-GARCH Approach Emphasizing Positive Definiteness," *FMDB Transactions on Sustainable Social Sciences Letters*, vol. 3, no. 1, pp. 9–18, 2025.

1. Introduction

The global energy market is greatly influenced by the dynamics of crude oil prices, which serve as key indicators of macroeconomic stability and investment performance in all oil-dependent economies. For a major oil-producing country like Nigeria, understanding the volatility and interconnectedness of various crude oil price parameters is critical for informed financial planning, risk management, and policymaking. Crude oil, as a globally traded commodity, plays a critical role in economic development, energy security, and fiscal and monetary policymaking, particularly in oil-dependent countries like

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Nigeria. Crude oil price volatility not only affects macroeconomic stability but also has implications for exchange rate dynamics, investment flows, and government revenue forecasts [7]. Recent global events, including geopolitical tensions, pandemic-induced disruptions, and shifting energy consumption patterns, have intensified the unpredictability of oil markets. In this context, understanding the dynamics of returns and volatility transmission mechanisms among major crude oil benchmarks—such as Brent, Dubai, West Texas Intermediate (WTI), and the average crude oil price—has become critical for policymakers, traders, and economic analysts. It is also necessary to absorb the effects of contagions and market shocks that cross borders and affect national incomes [19]. While oil price modelling has increased, many previous studies have been limited by methodological gaps, particularly in their ability to capture time-varying volatility and spillover effects in a multivariate framework.

Traditional linear models often fail to account for the complex interconnectedness and heterogeneous nature of crude oil price returns. Furthermore, previous multivariate GARCH models, such as the VECH representation, faced limitations related to the lack of a positive precision guarantee in conditional covariance matrices and excessive computational burden resulting from overparameterization. These shortcomings have important implications, as failure to maintain positive precision can weaken the validity of statistical inference and lead to an unstable risk model [18]. To address these methodological challenges, this study uses the diagonal BEKK-GARCH model. This sophisticated multivariate approach not only ensures the positive accuracy of the conditional covariance matrix but also reduces the dimensionality of the coefficients without compromising model accuracy. This modelling framework, developed by Engle and Kroner [11], provides a robust structure for capturing volatility clustering, dynamic correlations, and spillover effects among multiple crude oil price returns. In particular, the country-specific model specification simplifies the estimation process while preserving the underlying dynamics, making it particularly suitable for modelling highly correlated financial time series. This study uses a multivariate modelling framework to analyse the average monthly returns of four major US dollar-denominated crude oil benchmark indices per barrel: Brent (COB), Dubai (COD), West Texas Intermediate (COWTI), and Average Prices (COA). Data spanning from January 1982 to April 2023, obtained from the Central Bank of Nigeria (CBN) Statistical Database, 1984, provided a robust econometric estimate.

Preliminary diagnostics, including stationarity tests, volatility plots, descriptive statistics, and normality checks, were conducted to ensure that the data conformed to the assumptions required for higher-order multivariate analysis [7]. To effectively model the dynamic variance and conditional covariance structure of these parameters, the study adopts the BEKK-GARCH diagonal model developed by Engle and Kroner [11], with emphasis on ensuring identification of the conditional variance-covariance matrix. The diagonal representation, which simplifies coefficient estimation while maintaining model flexibility, facilitates capturing time-varying volatility and correlation across markets. Unlike the standard VECH-GARCH model, the BEKK diagonal formula ensures that the estimated conditional covariance matrix remains symmetric and positively defined, a prerequisite for modelling financial time series. This modelling approach also accounts for the presence of ARCH effects and large deviations from normality observed in the return data, as evidenced by high Jarque-Bera statistics. The return series of all benchmark indices displayed left-skewed distributions with high kurtosis, confirming the presence of volatility clustering and fat tails. Through this methodological lens, the study not only quantitatively quantifies the volatility implications among crude oil indices but also contributes to the literature by emphasising the importance of model structure and mathematical rigour (especially ensuring identification) in multivariate volatility modelling.

By applying the Diagonal BEKK-GARCH framework to crude oil prices, this research provides a statistically sound and policy-relevant analysis that deepens our understanding of market interconnectedness and provides energy sector actors with actionable insights to navigate global oil market uncertainty. The primary objective of this study is to explore the dynamic interrelationship among key crude oil price returns using a country-specific BEKK-GARCH model that ensures accuracy, reduces computational complexity, and provides precise volatility and variance estimates. Specifically, the study aims to examine the effects of agglomeration and volatility transmission on oil price parameters, assess the suitability of the country-specific BEKK-GARCH specification for modelling these dynamics, and demonstrate the statistical validity of positive precision in conditional covariance matrices through matrix decomposition and appropriate eigenvalue analysis. Through this approach, the study contributes to closing the empirical and methodological gaps in multivariate modelling of oil price returns, providing a more stable and interpretable framework for risk management and policymaking in oil-exporting economies.

2. Literature Review

Understanding the dynamics of crude oil price returns and volatility has received significant academic attention due to the central role of petroleum in the global economy. A large body of literature has used time series models to capture the underlying volatility, continuity, and shock transmission in oil price movements. Early studies, such as those by Bendick [8] and Hamilton [4], emphasised the stochastic nature of crude oil prices and the macroeconomic effects of crude oil price shocks. These seminal works emphasised the need for robust statistical models capable of handling time-varying volatility. With the development of autoregressive conditional heteroskedasticity (ARCH) models by Engel [12] and its extension to generalised ARCH (GARCH) by Bollerslev [15], researchers gained tools for modelling volatility clustering, a stylised reality of financial time series. While univariate GARCH models effectively capture the volatility of individual assets, they fail to account for the interdependence of multiple return series, which is particularly important for oil benchmarks traded in interconnected global markets. To address these limitations, multivariate GARCH models (MGARCH) were introduced. Among the various specifications, the BEKK-GARCH model proposed by Engle [12] stands out for its ability to ensure the positive accuracy of the conditional covariance matrix, which is a prerequisite for risk modelling and portfolio optimisation. The BEKK framework has been widely applied in energy economics to study the implications of volatility and co-movements between commodities and financial assets [5]; [17]; [9]. Recently, researchers have adopted the Diagonal BEKK-GARCH Model, which provides a cheaper alternative to the full BEKK model while preserving key statistical properties. The diagonal structure reduces the number of parameters, improving model usability and estimation efficiency, especially in higher-dimensional systems [1].

Studies by Hamouda et al. [10] and Reboredo [3] used the diagonal BEKK model to study volatility correlations between oil, gold, and stock markets, confirming the presence of large spillover effects and dynamic correlations. Specifically, in the context of oil markets, many researchers have analysed movements in oil benchmark prices. For example, Alawi and Mabrouk [14] explored volatility transmission between WTI, Brent, and OPEC benchmark crudes using MGARCH models and found a close correlation. Similarly, Kang et al. [16] examine the impact of global supply and demand shocks on oil price volatility across different benchmarks, highlighting the importance of capturing both common and unique shocks in multivariate settings. Despite these contributions, relatively few studies have explicitly emphasised the identification property of the covariance matrix. This property is crucial for ensuring the stability and validity of risk assessments. Ensuring identification is especially important when analysing the returns of multiple oil indices with different regional characteristics and geopolitical risk exposure. This study builds on these studies by using the BEKK-GARCH Diagonal framework to model the volatility dynamics of three major crude oil price indices: WTI, Brent, and Dubai. By focusing on the positive specificity of the covariance matrix, the study improves the robustness of multivariate volatility modelling in energy economics. It provides reliable insights into the structural interconnectedness of global oil markets.

3. Methodology

The data used for this study were obtained and extracted from the Energy Information Administration website. The variables consisted of the monthly average of crude oil (COA), Brent crude oil (C0B), Dubai crude oil (COD), WTI crude oil (COW) in dollars per Naira per Barrel ($/barrel), from January 1982 to April 2023. They make up a total of 1984 data points. Econometric view (EViews) version 10 is used for data analysis. A preliminary analysis was conducted to verify the reliability of the measures, assess the effectiveness of any analysis, examine the distribution of individual variables, identify anomalies, and ensure that the data fit the proposed model. Some preliminary analyses include logarithmic return and volatility, time plots, descriptive statistics, unit root tests for stationarity, ARCH effects, correlation analysis, unlimited VAR models, and optimal lag lengths. For logarithmic return and volatility, the Crude oil price is fitted to a conditionally compound monthly return computed as,

$$\mathrm{RCOA} = \mathrm{Log}\left(\frac{\mathrm{COA_t}}{\mathrm{COA_{t-1}}}\right) \mathrm{X} \frac{100}{1} \tag{1}$$

$$\mathrm{RCOB} = \mathrm{Log}\left(\frac{\mathrm{COB_t}}{\mathrm{COB_{t-1}}}\right) \mathrm{X} \frac{100}{1} \tag{2}$$

$$\mathrm{RCOD} = \mathrm{Log}\left(\frac{\mathrm{COD_t}}{\mathrm{COD_{t-1}}}\right) \mathrm{X} \frac{100}{1} \tag{3}$$

$$\mathrm{RCOWTI} = \mathrm{Log}\left(\frac{\mathrm{COWTI_t}}{\mathrm{COWTI_{t-1}}}\right) \mathrm{X} \frac{100}{1} \tag{4}$$

For t = 1, 2, ….t-j where COA_t, COB_t, COD_t and $COWTI_t$ represents the four types of crude oil price returns at time t, and COA_{t-1}, COB_{t-1}, COD_{t-1} and $COWTI_{t-1}$crude oil price at time "t-1". The transformation process of calculating returns on the price is performed to eliminate outliers, achieve stationarity, and reduce the volatility of the variables. The Jarque-Bera (JB) test, developed by Engle [12], is a statistical method used to assess whether a dataset follows a normal distribution by assessing its deviation and dispersion. If the test statistics are significantly different from zero, it indicates that the data does not have a normal distribution. When the dataset is not tested normally, alternative inferential statistics can be used, such as the multivariate GARCH model with specific assumptions about error distribution. To determine the stability of a time series, the Dickey-Fuller (ADF) test is used. The null hypothesis of the ADF test is that the time series has a unit root, indicating instability. If the test statistic is negative than the critical value, the null hypothesis is rejected, indicating that the data is constant.

In the time series analysis, it is crucial to determine the appropriate delay length using the VAR model. A vector autoregression model (VAR) is a statistical model used to capture the linear correlation between multiple time series. The optimal delay length of the VAR model can be determined using information criteria such as the Akaike Information Criterion (AIC), Schwarz Information Criterion (SIC), and Hannan-Quinn Information Criterion (HQ). When analysing multiple time series, it is important to assess whether they are integrated, meaning they have a long-term equilibrium relationship. The Johansen cointegration test is a method used to determine the number of integration relationships between variables. In the case of joint integration, the Vector Error Correction Model (VECM) can be used to model both short-term dynamics and long-term relationships between variables. According to Engle and Kroner [11], the word BEKK was named after Baba-Engle, Kraft, and Kroner, and was proposed basically to improve the VECM-GARCH model. The covariance matrix is a positive definite matrix, making the parameters easier to estimate by measuring the correlation and reflecting the direction of spillover effects. The model is given as:

$$\sigma_{i,t}^{2} = \mathrm{M}\,\mathrm{M}^{1} + \mathrm{A} * \varepsilon_{i,t-1}\,\varepsilon_{1,t-1}^{1} * \mathrm{A}^{1} + \mathrm{B} * \sigma_{i,t-1}\mathrm{B}^{1},\quad \Xi_{t}/\Psi_{t-1} \sim \mathrm{N}(0, \mathrm{H}_{t}) \tag{5}$$

M, A, and B are all N × N parameter matrices, and M is a lower triangular matrix. The decomposition of the constant term into a product of two triangular matrices is to ensure the positive definiteness of the covariance matrix. $\sigma_{i,t-1}$. A and B are diagonal matrices. Engle and Kroner [11] show that the BEKK model is covariance stationary if and only if the eigenvalues of $A \otimes A^{1} + B \otimes B^{1}$ are less than one in modulus, where ⊗ denotes the Kronecker product of two matrices. Whenever $K > 1$, an identification problem arises because several parameterisations yield the same representation of the model. Engle and Kroner [11] give conditions for eliminating redundant, observationally equivalent representations. This is called diagonal BEKK, proposed by Bollerslev [15]. The main advantage is that the number of parameters decreases to $N(N+1)/2+2N$ while still maintaining the positive definiteness of σ_t . Also, the Eviews representation of the BEKK model by Engle and Kroner [11] in matrix form is given as:

$$\mathrm{H}_{t} = \mathrm{W}^{1}\mathrm{W} + \mathrm{A}^{1}\mathrm{H}_{t-1}\mathrm{A} + \mathrm{B}^{1}\Xi_{t-1}\Xi_{t-1}^{1}\mathrm{B} \tag{6}$$

Also, the econometric views (Eviews) format for representing the estimate of a BEKK model by Engle and Kroner [11] is given as:

$$\mathrm{H}_{t} = \mathrm{W}^{1}\mathrm{W} + \mathrm{A}^{1}\mathrm{H}_{t-1}\mathrm{A} + \mathrm{B}^{1}\Xi_{t-1}\Xi_{t-1}^{1}\mathrm{B} \tag{7}$$

3.1. Variance Equation

$$\sigma_{1,t}^{2} = \mathrm{M} + \mathrm{A1}(i,j)^{2}\varepsilon_{i,t-1}^{2} + \mathrm{B1}(i,j)^{2}\sigma_{i,t-1}^{2}$$

Where $M > 0$ and M is a scalar, $A1(i,j)$ *and* $B1(i,j)$ are diagonal matrices.

$$\sigma_{1,t}^{2} = \begin{bmatrix} \mathrm{M} \\ \mathrm{M} \\ \mathrm{M} \\ \mathrm{M} \end{bmatrix} + \begin{bmatrix} \mathrm{A1}(1,1)^{2} \\ \mathrm{A1}(2,2)^{2} \\ \mathrm{A1}(3,3)^{2} \\ \mathrm{A1}(4,4)^{2} \end{bmatrix} \varepsilon_{i,t-1}^{2} + \begin{bmatrix} \mathrm{B1}(1,1)^{2} \\ \mathrm{B1}(2,2)^{2} \\ \mathrm{B1}(3,3)^{2} \\ \mathrm{B1}(4,4)^{2} \end{bmatrix} \sigma_{i,t-1}^{2}$$

$$\sigma_{1,t}^{2} = \mathrm{M} + \mathrm{A1}(1,1)^{2}\varepsilon_{1,t-1}^{2} + \mathrm{B1}(1,1)^{2}\sigma_{1,t-1}^{2}$$

$$\sigma_{2,t}^{2} = \mathrm{M} + \mathrm{A1}(2,2)^{2}\varepsilon_{1,t-1}^{2} + \mathrm{B1}(2,2)^{2}\sigma_{2,t-1}^{2}$$

$$\sigma_{3,t}^{2} = \mathrm{M} + \mathrm{A1}(3,3)^{2}\varepsilon_{3,t-1}^{2} + \mathrm{B1}(3,3)^{2}\sigma_{3,t-1}^{2}$$

$$\sigma_{4,t}^{2} = \mathrm{M} + \mathrm{A1}(4,4)^{2}\varepsilon_{4,t-1}^{2} + \mathrm{B1}(4,4)^{2}\sigma_{1,t-1}^{2}$$

3.2. Covariance Equation

$$\rho_{1,2,t} = \mathrm{M} + \mathrm{A1}(1,1)\varepsilon_{1,t-1} * \varepsilon_{2,t-1} + \mathrm{B1}(1,1)\rho_{1,2,t-1}$$

$$\rho_{1,3,t} = \mathrm{M} + \mathrm{A1}(1,3)\varepsilon_{1,t-1} * \varepsilon_{3,t-1} + \mathrm{B1}(1,3)\rho_{1,3,t-1}$$

$$\rho_{1,4,t} = \mathrm{M} + \mathrm{A1}(1,4)\varepsilon_{1,t-1} * \varepsilon_{4,t-1} + \mathrm{B1}(1,4)\rho_{1,3,t-1}$$

$$\rho_{2,3,t} = \mathrm{M} + \mathrm{A1}(2,3)\varepsilon_{2,t-1} * \varepsilon_{3,t-1} + \mathrm{B1}(2,3)\rho_{1,3,t-1}$$

$\rho_{2,4,t} = M + A1(2,4)\varepsilon_{2,t-1} * \varepsilon_{4,t-1} + B1(2,4)\rho_{2,4,t-1}$

$\rho_{3,4,t} = M + A1(3,4)\varepsilon_{3,t-1} * \varepsilon_{4,t-1} + B1(3,4)\rho_{3,4,t-1}$

Where $M > 0$ and M is a scalar, $A1(i,j)$ *and* $B1(i,j)$ are diagonal matrices, also, to prove the positive definiteness of the matrices $M, A1$ *and* $B1$, we need to show that for any nonzero vector, the quadratic form satisfies: $X^T Mx > 0, X^T A1x > 0,$ *and* $X^T B1x > 0.$ For all $X \neq 0$. A matrix is positive definite if all its eigenvalues are positive, or, equivalently, if its leading principal minors (determinants of upper-left submatrices) are positive. Since A1 and B1 are diagonal matrices with strictly positive diagonal elements, they are trivially positive definite. This follows from the fact that for a diagonal matrix, all its eigenvalues are its diagonal elements, which are all positive. Thus, A1 and B1 satisfy the conditions:

$X^T A1x > 0, \sum_{i=1}^{3} A_{ii}\, x_i^2 > 0;\ \forall x \neq 0$

$X^T B1x > 0,\quad \sum_{i=1}^{3} B_{ii}\, x_i^2 > 0;\ \forall x \neq$ zero where $A_{ii.}\ B_{ii} > 0,$ meaning both matrices are positive definite.

4. Results

Statistics for both crude oil and yield benchmarks, the unit test for crude oil price benchmarks, the plots for the return of the crude oil price (naira/dollar) for four major benchmarks for the crude oil price, First, we present Figure 1, plot of crude oil price (naira/dollar) data in four major crude oil benchmarks such as West Texas Intermediate (WTI), Brent Blend, Dubai Crude and the average. The results are shown as follows.

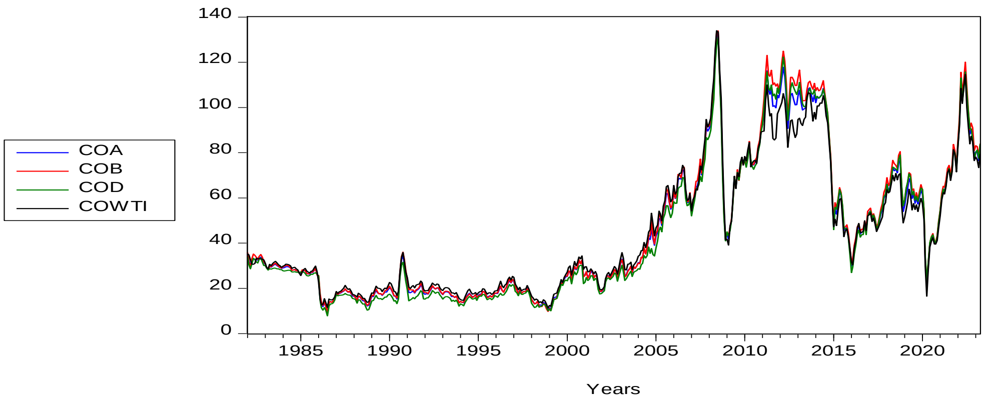

Figure 1: Plots of the raw data on crude oil price (naira/dollar) in four major crude oil benchmarks, such as the average, Brent Blend, Dubai crude and West Texas Intermediate (WTI)

Table 1 contains the results of descriptive statistics of the raw and returns on the Nigerian Crude oil price. This was carried out to know whether the returns on price benchmark obey the normality assumption, and it is estimated using the Jarque-Bera test. The results show that that Mean of all series; the average(COA(44.885)), Brent Blend(COB(46.0149)), Dubai Crude(COD(43.899)) and West Texas Intermediate (COWTI (44.741)) and their returns series such returns on the average(RCOA(0.173)), Brent Blend(RCOB(0.172)), Dubai Crude(RCOD(0.182)) and West Texas Intermediate (FCOWTI (0.164)) all have positive mean. Similarly, the results of Skewness statistics show that for the raw series, average(COA(0.890)), Brent Blend(COB(0.9149)), Dubai Crude(COD(0.903)), and West Texas Intermediate (COWTI (0.876)) are all positive. This means that the raw series are all skewed to the right, while their returns series, such as returns on the average (RCOA (-0.677)), Brent Blend (RCOB (-0.549)), Dubai Crude (RCOD (-0.707)) and West Texas Intermediate (RCOWTI (-0.676)) series are all skewed to the left.

Table 1: Descriptive statistics on raw and return on crude oil price benchmarks

	COA	**COB**	**COD**	**COWTI**	**RCOA**	**RCOB**	**RCOD**	**RCOWTI**
Mean	44.885	46.0149	43.899	44.741	0.173	0.172	0.182	0.164
Median	30.700	30.975	28.950	31.730	0.902	0.565	0.866	0.801
Maximum	132.830	133.870	131.220	133.930	43.020	43.263	49.102	54.744
Minimum	9.620	9.450	7.850	11.310	-50.491	-51.143	-54.012	-59.262

Std. Dev.	30.395	31.741	31.162	28.463	9.163	9.3701	9.474	9.411
Skewness	0.890	0.9149	0.903	0.876	-0.677	-0.549	-0.707	-0.676
Kurtosis	2.603	2.657	2.612	2.648	8.313	6.859	9.274	11.028
Jarque-Bera	68.692	71.623	70.548	65.952	620.109	331.795	853.059	1367.02
Probability	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
Sum	22263.08	22823.37	21774.1	22191.62	85.554	85.278	90.243	81.112
Sum Sq.D	457293.6	498712.3	480674	401026.6	41482.	43372.7	44335.7	43753.3
Obsers	496	496	496	496	495	495	495	495

This means that the returns series are all negatively skewed. Also, Kurtosis indicates how much data resides in the tails. The Kurtosis of all the raw series; the average (COA(2.603)), Brent Blend(COB(2.657)), Dubai Crude(COD(2.612)) and West Texas Intermediate (COWTI (2.648)) are less than 3, whereas that of the returns series such returns on the average(RCOA(8.313)), Brent Blend(RCOB(6.859)), Dubai Crude(RCOD(9.274)) and West Texas Intermediate (COWTI (11.028)) all greater than 3. The kurtosis greater than 3 indicates a leptokurtic shape. The data set is concentrated at the peak, and there are a lot of outliers, whereas a decreased kurtosis corresponds to a broadening of the peak and "thickening" of the tails. The data distribution has a pointy peak. Similarly, the Jarque–Bera test is a goodness-of-fit test to determine whether data samples have the skewness and kurtosis matching a normal distribution.

The results of the Jarque-Bera statistics show that for the raw series; average (COA (68.692)), Brent Blend(COB(71.623)), Dubai Crude(COD(70.548)) and West Texas Intermediate (COWTI (65.952)) whereas the returns series such returns on the average(RCOA((620.109)), Brent Blend(RCOB(331.795)), Dubai Crude(RCOD(853.059)) and West Texas Intermediate (RCOWTI ((1367.020)). Since all the p-values are smaller than 0.05, we reject the null hypothesis and conclude at a 95% confidence level that all the variables are indeed not significant. In line with this effect, Abdulkarem et al. [6] suggested that alternative inferential statistics, such as those involving error distribution assumptions with fixed degrees of freedom, will be integrated into the ARCH, GARCH, or multivariate GARCH model, necessitating a selection process. This is followed by the unit Root using the Augmented Dickey-Fuller statistics test. The results are shown in Table 2 below.

Table 2: Unit test on raw crude oil price benchmarks

Variable	t-Statistic	P-Value	Remarks
COA	-2.220	0.199	**1(1)**
D(COA)	-14.976	0.000	
COB	-2.172	0.217	**1(1)**
D(COB)	-15.415	0.000	
COD	-2.171	0.217	**1(1)**
D(COD)	-14.637	0.000	
COWTI	-2.367	0.152	**1(1)**
D(COWTI)	-15.396	0.000	

The results of the unit test on raw and return on Crude Oil Price Benchmarks are shown in Table 2. The variables are stationary at first difference, and the time plot for the stationary series is shown below.

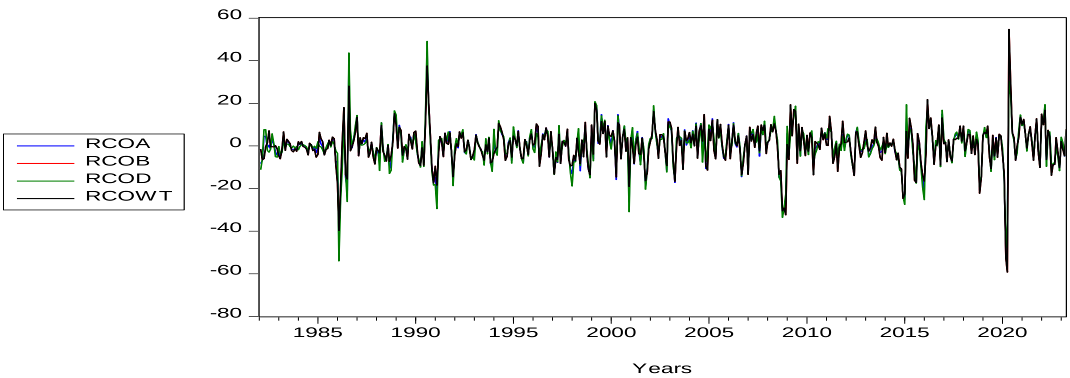

Figure 2: Plots of the returns on crude oil price (naira/dollar) in four major crude oil benchmarks, such as the average, Brent Blend, Dubai crude and West Texas Intermediate (WTI)

Figure 2 is a plot of the returns on the crude oil prices Average, Brent, Dubai and West Texas Intermediate from January 1982 to May 2023. From visual examination, the behaviour of crude oil prices shows that Figure 2 is stationary as the series fluctuates around the origin. It also reveals the presence of clustering volatilities, following Orakau [2], Kanchan et al. [13], Deebom and Tuaneh [18], Deebom et al. [19], and Ali et al. [7] method for using Multivariate GARCH Models in modelling multiple time series. The results of the Diagonal Baba-Engle -Kraft-Kroner (BEKK) model are shown in both matrix and equation forms below.

$$\mathrm{M} = \begin{bmatrix} 35.167(0.000) & 34.801(0.000) & 36.847(0.000) & 34.447(0.00) \\ & 34.949(0.000) & 36.403(0.000) & 33.403(0.000) \\ & & 39.381(0.000) & 35.423(0.000) \\ & & & 34.850(0.000) \end{bmatrix},$$

$$\mathrm{A1} = \begin{bmatrix} 0.602(0.000) & & & \\ & 0.578(0.000) & & \\ & & 0.621(0.000) & \\ & & & 0.620(0.000) \end{bmatrix},$$

$$\mathrm{B1} = \begin{bmatrix} 0.652(0.000) & & & \\ & 0.681(0.000) & & \\ & & 0.622(0.000) & \\ & & & 0.640(0.000) \end{bmatrix}$$

4.1. Variance Equation

$$\sigma^2_{1,t} = \mathrm{M} + \mathrm{A1(i,j)}^2\varepsilon^2_{i,t-1} + \mathrm{B1(i,j)}^2\sigma^2_{i,t-1}$$

Where $M > 0$ and M is a scalar, $A1(i,j)$ *and* $B1(i,j)$ are diagonal matrices.

$$\sigma^2_{1,t} = \begin{bmatrix} 35.167 \\ 34.948 \\ 39.381 \\ 34.850 \end{bmatrix} + \begin{bmatrix} 0.362 \\ 0.334 \\ 0.386 \\ 0.384 \end{bmatrix}\varepsilon^2_{i,t-1} + \begin{bmatrix} 0.425 \\ 0.464 \\ 0.387 \\ 0.410 \end{bmatrix}\sigma^2_{i,t-1}$$

4.2. Covariance Equation

$$\rho_{1,2,t} = 34.801 + 0.348\varepsilon_{1,t-1} * \varepsilon_{2,t-1} + 0.444\rho_{1,2,t-1}$$

$$\rho_{1,3,t} = 36.847 + 0.374\varepsilon_{1,t-1} * \varepsilon_{3,t-1} + 0.405\rho_{1,3,t-1}$$

$$\rho_{1,4,t} = 34.447 + 0.373\varepsilon_{1,t-1} * \varepsilon_{4,t-1} + 0.417\rho_{1,3,t-1}$$

$$\rho_{2,3,t} = 36.403 + 0.359\varepsilon_{2,t-1} * \varepsilon_{3,t-1} + 0.424\rho_{1,3,t-1}$$

$$\rho_{2,4,t} = 33.617 + 0.359\varepsilon_{2,t-1} * \varepsilon_{4,t-1} + 0.436\rho_{2,4,t-1}$$

$$\rho_{3,4,t} = 35.423 + 0.385\varepsilon_{3,t-1} * \varepsilon_{4,t-1} + 0.398\rho_{3,4,t-1}$$

To verify the positive definiteness of M1, we compute its leading principal minors (determinants of upper-left submatrices). We considered; the first leading principal minor (1×1 determinant) gives det(M1(1,1) =35.167>0. Similarly, the Second leading principal minor (2×2 determinant):

$$\det(\mathrm{M}_2) = \begin{bmatrix} 35.167 & 34.801 \\ 34.801 & 34.949 \end{bmatrix} = (35.167 \times 34.949) - (34.801)^2$$

=1,2228.922−1210.293=18.629 >0

Third leading principal minor (3×3 determinant, full determinant of M):

$$\det\left(\begin{bmatrix}35.167 & 34.801 & 36.847\\ 34.801 & 34.949 & 36.403\\ 36.847 & 36.403 & 39.381\end{bmatrix}\right)$$

$$= 35.167\begin{bmatrix}34.949 & 36.403\\ 36.403 & 39.381\end{bmatrix} - 34.801\begin{bmatrix}34.801 & 36.403\\ 36.847 & 39.381\end{bmatrix} + 36.847\begin{bmatrix}34.801 & 34.949\\ 36.847 & 36.403\end{bmatrix}$$

$$\det([\mathrm{M}(3\mathrm{x}3]) > 0$$

The analysis of volatility series within the diagonal BEKK model is conducted to investigate the behaviour of the conditional variance and conditional covariance. This then provides evidence of dynamic and time-varying co-movements between the logarithmic returns of the variables. The variables for the study are the returns on crude oil price indices such as Average, Brent, Dubai, and West Texas Intermediate (WTI). The matrix M in the model captures the constant component of the covariance, reflecting the baseline level of volatility and covariances among these variables. The elements M (i, j) and M (i, j) represent the constant covariance between the ith and j-th row and column matrix. For instance, M (1,1) (35.167) indicates the baseline variance of the average crude oil price returns, while M (1,2) (34.801) represents the baseline covariance between the Average and Brent crude oil price returns. All reported M (i, j) estimates are statistically significant with p-values of 0.0000, suggesting that these baseline covariances are robust. The diagonal elements of matrix A1 measure the sensitivity of current volatility to past squared residuals (innovations) for each crude oil price return. Specifically, A1(1,1) (0.602) implies that approximately 60.2% of past shocks in the Average crude price returns are transmitted to its current volatility. Similarly, A1(2,2) (0.578), A1(3,3) (0.621) and A1(4,4) (0.620) indicate that past shocks in Brent, Dubai, and WTI returns have significant effects on their current volatilities, with all estimates being highly significant (p-values of 0.0000).

This underscores the presence of strong ARCH effects in these return series. The diagonal elements of matrix B1 capture the persistence of volatility over time, reflecting the influence of past volatility on current volatility. For example, B1(1,1) (0.652), B1(2,2) (0.681), B1(3,3) (0.622), and B1(4,4) (0.6400) suggest that about 65.2%, 68.1%, 62.2% and 64.00% of past volatility in the Average crude returns carries over to the present period. This indicates substantial volatility persistence in Brent, Dubai, and WTI returns, respectively, with all estimates being statistically significant (p-values of 0.0000). This highlights the enduring nature of volatility in these markets. The significance of the A1 and B1 parameters has important economic implications. The pronounced ARCH effects (A1) suggest that recent shocks to crude oil returns have a strong impact on current volatility, indicating that these markets are highly sensitive to new information. The substantial GARCH effects (B1) imply that volatility is persistent, meaning that periods of high volatility are likely to be followed by continued high volatility, and similarly for low volatility periods. This persistence can affect investment decisions, risk management, and policy formulations in the energy sector. The results of this study are consistent with those of Kanchan et al. [13], who conducted a study on volatility spillover using a multivariate GARCH model applied to futures and spot market prices of black pepper.

In Kanchan et al. [13], it was found that the futures market is the main transmitter of volatility into the spot market, resulting in higher persistence in the volatility of the black pepper spot market. Analysis of volatility spillover in this context also revealed bidirectional volatility spillover between the spot and futures markets of black pepper. For positive definiteness, the above illustrations show that leading principal minors up to this point are positive. Given the symmetry and positive definiteness of the submatrices, we can infer that the matrix M is positive definite. Therefore, based on Sylvester's criterion and the positive values of the leading principal minors, the matrix M is positive definite. This characteristic is crucial for the validity of the model, as it ensures that the estimated covariances are meaningful and reliable. The Diagonal BEKK model applied to the returns of Average, Brent, Dubai, and WTI crude oil prices reveals significant ARCH and GARCH effects, indicating that both past shocks and volatility strongly influence current volatility. The model's structure ensures the positive definiteness of the covariance matrix, providing a robust framework for understanding the volatility dynamics of these crude oil markets.

Table 3: Estimation results for portmanteau tests

Lags	Q-Stat	Prob.	Adj Q-Stat	Prob.	Df
1	22.93649	0.1154	23.05721	0.1122	16
2	46.25668	0.0494	46.62417	0.0458	32
3	59.27973	0.1274	59.85504	0.1171	48
4	91.70694	0.0132	92.97588	0.0105	64
5	100.8715	0.0574	102.3868	0.0466	80
6	113.1738	0.1113	115.0881	0.0896	96
7	122.4939	0.2343	124.7627	0.1931	112
8	155.4524	0.0497	159.1621	0.0322	128
9	172.4273	0.0532	176.9764	0.0321	144

10	192.1867	0.0420	197.8274	0.0225	160
11	204.9934	0.0664	211.4168	0.0352	176
12	223.8200	0.0576	231.5055	0.0271	192

Model diagnostic checks were conducted to determine the adequacy of the estimated model, as shown in Table 3 for the estimation results of portmanteau tests and in Figure 3 for the results. The model diagnostic check revealed the BEKK GARCH as the most adequate model among the three competing models. The result obtained using this criterion was in line with Deebom and Tuaneh's [18] findings. Deebom and Tuaneh [18] model the exchange rate and Nigerian deposit money market dynamics using a trivariate form of the multivariate GARCH model. It was found that the diagonal MGARCH–BEKK estimation confirmed it was the best-fitted and appropriate model for modelling the exchange rate and Nigerian deposit money market dynamics using a trivariate form of multivariate GARCH model. Also, the test for conditional heteroscedasticity using the portmanteau test is shown in Table 3.

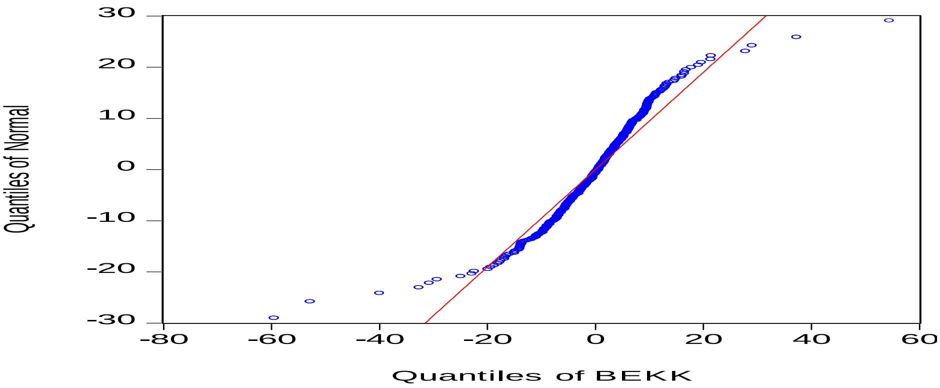

Figure 3: QQ Plot on the BEKK model to determine the adequacy of the model

The results obtained show that the null hypothesis of no residual autocorrelations should be accepted, while the alternative should be rejected. This means there is no conditional heteroscedasticity. System Residual Normality Tests using orthogonalisation cholesky (Lutkepohl) compare the null hypothesis that residuals are multivariate normal to the alternative hypothesis that residuals are not multivariate normal. The results obtained show that residuals are not multivariate normally distributed. This means that the null hypothesis should be rejected, while the alternative should be accepted. Similarly, the residual test for autocorrelations up to lag h using the Portmanteau Test is shown in Table 3. The estimation results for portmanteau tests show that there is an absence of autocorrelations up to lag h. Also, Figure 3 contains the QQ Plot on the BEKK model to determine the adequacy of the model. The results obtained with the procedures of Orakau [2], Kanchan et al. [13], Deebom and Tuaneh [18], Deebom et al. [19], and Ali et al. [7] for using Multivariate GARCH Models in modelling volatility properties in time series variables.

5. Conclusion

Empirical results demonstrate the existence of nonlinear volatility dynamics in crude oil price indices. Descriptive and statistical diagnostics confirmed that the return series exhibits abnormal properties, requiring effective modelling techniques. The estimation results indicate that both short-term shocks and long-term volatility persistence significantly influence the volatility dynamics of the returns on crude oil prices. The significant constant covariances suggest interconnectedness among the different crude oil markets. These insights are vital for market participants in formulating hedging strategies and for policymakers in understanding market stability. The diagonal BEKK-GARCH model successfully captures volatility clusters and spillover effects among oil indices, highlighting their interconnectedness.

The results of the country's BEKK-GARCH model, particularly the confirmation of the positive definiteness of the variance-covariance matrices (M, A1, and B1), have important policy implications for Nigeria's economic planning, oil price risk management, and energy market stability: enhancing stability in forecasting and risk assessment, improving the design of oil revenue management instruments, evidence-based pricing and benchmarking policies, and supporting energy diversification policies. The mathematical confirmation of the BEKK-GARCH model's accuracy ensures the statistical integrity and practical applicability of the volatility and correlation structures. This strengthens the evidence base for dynamic, risk-informed, and forward-looking fiscal and energy policies in crude oil markets, improving their resilience to global oil price shocks. These findings are vital for policymakers, investors, and stakeholders in the energy economy, as they provide insights for managing price risks and macroeconomic exposure to oil market volatility.

Acknowledgement: The author gratefully appreciates the Rivers State Universal Basic Education Board for their insightful guidance, cooperation, and unwavering support that significantly enriched the quality and outcome of this research work.

Data Availability Statement: The data supporting the results of this study can be obtained from the corresponding author upon a reasonable request and with appropriate justification.

Funding Statement: This study was carried out without any financial assistance or grant from governmental, commercial, or non-profit funding bodies.

Conflicts of Interest Statement: The author affirms that there are no conflicts of interest, financial or personal, that could have influenced the research findings or interpretations presented in this paper.

Ethics and Consent Statement: The research adhered strictly to ethical standards, ensuring voluntary participation, informed consent, and complete confidentiality of participant information throughout the study.

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